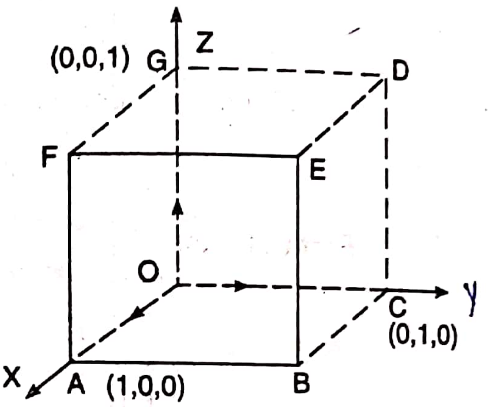
Example 116. *Verify Divergence Theorem, given that* $\overline{F} = 4xzi - y^2j + yzk$ *and S is the surface of the cube bounded by the planes x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.*

Solution. $\overline{\nabla} \cdot \overline{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot (4xzi - y^2j + yzk)$

$= 4z - 2y + y$

$= 4z - y$



Volume Integral $= \iiint \nabla \cdot F \, dv$

$= \iiint (4z - y) \, dx \, dy \, dz$

$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y) \, dz$

$= \int_0^1 dx \int_0^1 dy \, (2z^2 - yz)_0^1 = \int_0^1 dx \int_0^1 dy \, (2 - y)$

$$= \int_0^1 dx \left(2y - \frac{y^2}{2} \right)_0^1 = \int_0^1 dx \left(2 - \frac{1}{2} \right) = \frac{3}{2} \int_0^1 dx = \frac{3}{2} (x)_0^1 = \frac{3}{2} \quad \text{...(1)}$$

To evaluate $\iint_S F \cdot \hat{\eta} \, ds$ where S consists of six plane surfaces.

Over the face $OABC$, $z = 0$, $dz = 0$, $\hat{\eta} = -k$, $ds = dx \, dy$

$$\iint F \cdot \hat{\eta} \, ds = \int_0^1 \int_0^1 (-y^2 j) \cdot (-k) \, dx \, dy = 0$$

Over the face $BCDE$, $y = 1$, $dy = 0$, $\hat{\eta} = j$, $ds = dx \, dz$

$$\iint F \cdot \hat{\eta}\, ds = \int_0^1 \int_0^1 (4xzi - j + zk) \cdot (j)\, dx\, dz = \int_0^1 \int_0^1 -dxdz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1 (z)_0^1 = -(1)(1) = -1$$

Over the face $DEFG$, $z = 1, dz = 0, \hat{\eta} = k, ds = dx\, dy$

$$\iint F \cdot \hat{\eta}\, ds = \int_0^1 \int_0^1 [4x(1) - y^2 j + y(1)k] \cdot (k)\, dx\, dy$$

$$= \int_0^1 \int_0^1 y\, dx\, dy = \int_0^1 dx \int_0^1 y\, dy = (x)_0^1 \left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Over the face $OCDG, x = 0, dx = 0, \hat{\eta} = -i, ds = dy\, dz$

$$\iint F \cdot \hat{\eta}\, ds = \int_0^1 \int_0^1 (oi - y^2 j + yzk) \cdot (-i)\, dy\, dz = 0$$

Over the face $AOGF, y = 0, dy = 0, \hat{\eta} = -j, ds = dx\, dz$

$$\iint F \cdot \hat{\eta}\, ds = \int_0^1 \int_0^1 (4xzi) \cdot (-j)\, dx\, dz = 0$$

Over the face $ABEF, x = 1, dx = 0, \hat{\eta} = i, ds = dy\, dz$

$$\iint F \cdot \hat{\eta}\, ds = \int_0^1 \int_0^1 [(4zi - y^2 j + yzk) \cdot (i)]\, dy\, dz = \int_0^1 \int_0^1 4z\, dy\, dz$$

$$= \int_0^1 dy \int_0^1 4z\, dz = \int_0^1 dy\, (2z^2)_0^1 = 2\int_0^1 dy$$

$$= 2(y)_0^1 = 2$$

On adding we see that over the whole surface

$$\iint F \cdot \hat{\eta}\, ds = \left(0 - 1 + \frac{1}{2} + 0 + 0 + 2\right) = \frac{3}{2} \quad \text{...(2)}$$

From (1) and (2)

$$\iiint_V \nabla \cdot F\, dv = \iint_S \overline{F} \cdot \hat{\eta}\, ds$$ **Verified.**

Example 8·25. *Evaluate* $\int_S \mathbf{F} \cdot d\mathbf{s}$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$. (Kerala, 1995)

Sol. By divergence theorem,

$$\int_S \mathbf{F} \cdot d\mathbf{s} = \int_V div\, \mathbf{F}\, dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$

$$= \iiint_V (4 - 4y + 2z)\, dxdydz$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left|4z - 4yz + z^2\right|_0^3 dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\, dydx$$

$$= \int_{-2}^{2} \left|21y - 6y^2\right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4-x^2)}\, dx = 84 \int_0^2 \sqrt{(4-x^2)}\, dx$$

$$= 84 \left|\frac{x\sqrt{(4-x^2)}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right|_0^2 = 84\pi.$$

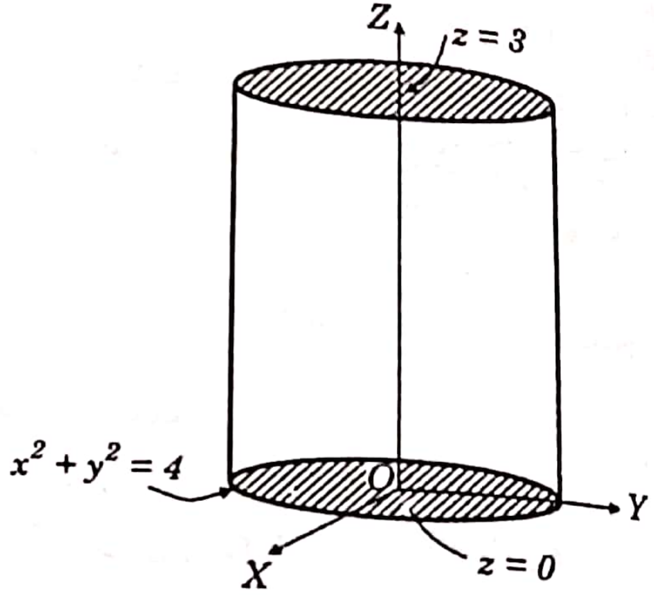


Fig. 8·20.

Ex. 34. *Find* $\iint_S \mathbf{A} \cdot \mathbf{n}\, dS$,

where $\mathbf{A} = (2x + 3z)\mathbf{i} - (xz + y)\mathbf{j} + (y^2 + 2z)\mathbf{k}$

and S is the surface of the sphere having centre at $(3, -1, 2)$ *and radius* 3. **(Kakatiya 1990, Meerut 74)**

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{A} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{A}\, dV.$$

Now $\operatorname{div} \mathbf{A} = \dfrac{\partial}{\partial x}(2x + 3z) + \dfrac{\partial}{\partial y}\{-(xz + y)\} + \dfrac{\partial}{\partial z}(y^2 + 2z)$

$= 2 - 1 + 2 = 3.$

$$\therefore \iint_S \mathbf{A} \cdot \mathbf{n}\, dS = \iiint_V 3\, dV = 3 \iiint_V dV = 3V.$$

But V is the volume of a sphere of radius 3. Therefore

$V = \frac{4}{3}\pi (3)^3 = 36\pi.$

$$\therefore \iint_S \mathbf{A} \cdot \mathbf{n}\, dS = 3V = 3 \times 36\pi = 108\pi.$$

Ex. 35. (a). *Apply divergence theorem to evaluate*

$$\iint_S [(x + z)\, dy\, dz + (y + z)\, dz\, dx + (x + y)\, dx\, dy]$$

Sol. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x + z) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(x + y)\right] dV$$

$= \iiint_V 2\, dV = 2 \iiint_V dV = 2V$, where V is the volume of the sphere $x^2 + y^2 + z^2 = 4$

$= 2\left[\dfrac{4}{3}\pi (2)^3\right] = \dfrac{64}{3}\pi.$

Ex. 35. (b). *By using the Gauss divergence theorem evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 4.$

(Osmania 1991)

Sol. By Gauss divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV,$$ where V is the volume enclosed by the sphere $x^2 + y^2 + z^2 = 4$

$$= \iiint_V (1 + 1 + 1)\,dV = 3\iiint_V dV = 3V = 3\cdot\left[\frac{4}{3}\pi(2)^3\right]$$

$$= 32\pi.$$

Ex. 36. *If S is any closed surface enclosing a volume V and* $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, *prove that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 6V.$$

(Rohilkhand 1980, Kanpur 79, Agra 78)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{F}\,dV = \iiint_V \text{div}\,(x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k})\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= \iiint_V (1 + 2 + 3)\,dV = 6\iiint_V dV = 6V.$$

Ex. 37. *Evaluate*

$$\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\cdot\mathbf{n}\,dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.* **(Agra 1969, Bombay 66)**

Sol. By divergence theorem, we have

$$\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\cdot\mathbf{n}\,dS$$

$$= \iiint_V \text{div}\,(y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\,dV,$$

where V is the volume enclosed by S

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\,dV = 2\iiint_V zy^2\,dV.$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate this triple integral. In polars $dV = (dr)(r d\theta)(r \sin\theta \, d\phi) = r^2 \sin\theta \, dr \, d\theta \, d\phi$. Also $z = r\cos\theta$, $y = r \sin\theta \sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π. The triple integral is

$$= 2 \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (r\cos\theta)(r^2 \sin^2\theta \sin^2\phi) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2 \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^5 \sin^3\theta \cos\theta \sin^2\phi \, dr \, d\theta \, d\phi$$

$$= 2 \cdot \frac{1}{6} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin^3\theta \cos\theta \sin^2\phi \, d\theta \, d\phi,$$

on integrating with respect to r.

[Note that the order of integration is immaterial because the limits of r, θ and ϕ are all constants].

$$= \frac{1}{3} \cdot \frac{2}{4 \cdot 2} \int_0^{2\pi} \sin^2\phi \, d\phi, \text{ on integrating with respect to } \theta$$

$$= \frac{1}{12} \cdot 4 \int_0^{\pi/2} \sin^2\phi \, d\phi = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{12}.$$

Ex. 38. *By converting the surface integral into a volume integral evaluate*

$$\iint_S (x^3 \, dy \, dz + y^3 \, dz \, dx + z^3 \, dx \, dy),$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. **(Bombay 1970)**

Sol. By divergence theorem, we have

$$\iint_S (F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz,$$

where V is the volume enclosed by S.

Here $F_1 = x^3$, $F_2 = y^3$, $F_3 = z^3$.

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3(x^2 + y^2 + z^2).$$

$\therefore$ the given surface integral

$$= \iiint_V 3(x^2 + y^2 + z^2) \, dx \, dy \, dz$$

Ex. 48. *Verify divergence theorem for the function* $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ *over the cylindrical region bounded by* $x^2 + y^2 = a^2$, $z = 0$ *and* $z = h$. (Kanpur 1975; Allahabad 79)

Sol. Let S denote the closed surface bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$, $z = h$. Also let V be the volume bounded by the surface S. By Gauss divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \text{div } \mathbf{F} \, dV.$$

We have $\iiint_V \text{div } \mathbf{F} \, dV = \iiint_V [\text{div } (y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k})] \, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)\right] dV = \iiint_V 2z \, dV$$

$$= 2\int_{z=0}^{h} \int_{x=-a}^{a} \int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} z \, dz \, dx \, dy$$

$$= 4\int_{z=0}^{h} \int_{x=-a}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} z \, dz \, dx \, dy$$

$$= 4\int_{z=0}^{h} \int_{x=-a}^{a} z \left[y\right]_{y=0}^{\sqrt{(a^2-x^2)}} dz \, dx$$

$$= 4\int_{z=0}^{h} \int_{x=-a}^{a} z \sqrt{(a^2 - x^2)} \, dz \, dx$$

$$= 8\int_{z=0}^{h} \int_{x=0}^{a} z \sqrt{(a^2 - x^2)} \, dz \, dx$$

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$$= 8 \int_{x=0}^{a} \sqrt{(a^2 - x^2)} \left[\frac{z^2}{2}\right]_{z=0}^{h} dx$$

$$= 8 \int_0^a \frac{h^2}{2} \sqrt{(a^2 - x^2)}\, dx = 4h^2 \int_0^a \sqrt{(a^2 - x^2)}\, dx$$

$$= 4h^2 \left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4h^2 \left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right]$$

$$= \pi a^2 h^2. \qquad \ldots (1)$$

Now we shall evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

The surface S consists of three surfaces: (i) the surface S_1 of the base of the cylinder *i.e.*, the plane face $z = 0$, (ii) the surface S_2 of the top face of the cylinder *i.e.*, the plane face $z = h$ and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 *i.e.*, $z = 0$, $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$, putting $z = 0$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_1 is obviously $-\mathbf{k}$.

$$\therefore \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (y\,\mathbf{i} + x\,\mathbf{j}) \cdot (-\mathbf{k})\, dS = 0.$$

For the surface S_2 *i.e.*, $z = h$, $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + h^2\,\mathbf{k}$, putting $z = h$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_2 is given by $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} (y\,\mathbf{i} + x\,\mathbf{j} + h^2\,\mathbf{k}) \cdot \mathbf{k}\, dS$$

$$= \iint_{S_2} h^2\, dS = h^2 \iint_{S_2} dS = h^2 .\ \text{area of the plane face } S_2 \text{ of the cylinder}$$

$$= h^2 . \pi a^2 = \pi a^2 h^2.$$

For the convex portion S_3 *i.e.*, $x^2 + y^2 = a^2$, a vector normal to S_3 is given by

$$\nabla (x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

$\therefore$ n = a unit vector along outward drawn normal at any point of S_3

$$= \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\mathbf{i} + y\mathbf{j}}{a}, \text{ since } x^2 + y^2 = a^2 \text{ on } S_3.$$

$$\therefore \text{ on } S_3, \mathbf{F}\cdot\mathbf{n} = (y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k})\cdot\left[\frac{1}{a}(x\mathbf{i} + y\mathbf{j})\right]$$

$$= \frac{1}{a}xy + \frac{1}{a}xy = \frac{2}{a}xy.$$

Also dS = elementary area on the surface S_3

= $a\, d\theta\, dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_3} \frac{2}{a}xy\, a\, d\theta\, dz, \text{ where}$$

$$x = a\cos\theta,\ y = a\sin\theta$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} 2a\cos\theta\, a\sin\theta\, d\theta\, dz$$

$$= 2a^2\int_{\theta=0}^{2\pi} \cos\theta\sin\theta\left[z\right]_{z=0}^{h} d\theta$$

$$= 2a^2h\int_0^{2\pi} \cos\theta\sin\theta\, d\theta = a^2h\int_0^{2\pi}\sin 2\theta\, d\theta$$

$$= a^2h\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} = -\frac{a^2h}{2}[\cos 4\pi - \cos 0] = 0.$$

Hence the total surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = 0 + \pi a^2h^2 + 0 = \pi a^2h^2. \quad \dots (2)$$

From (1) and (2), we see that

$$\iiint_V \text{div}\,\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

This verifies divergence theorem.